SYDNEY GRAMMAR SCHOOL

2012 Half-Yearly Examination

FORM VI
MATHEMATICS EXTENSION 1

Monday 27th February 2012

General Instructions

- Writing time — 2 Hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 70 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.

Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Collection

Section I Questions 1–10

- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.

Section II Questions 11–14

- Start each of these questions in a new booklet.
- Write your name, class and master clearly on each booklet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question Eleven.

Checklist

- SGS booklets — 4 per boy
- Candidature — 128 boys

Examiner
MLS

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

Question One

If $f(x) = \dfrac{x-1}{x}$, which of the following is equal to $f\left(\dfrac{1}{a}\right)$? **1**

(A) $1 - a$

(B) $\dfrac{a}{a-1}$

(C) $1 + a$

(D) $\dfrac{a-1}{a}$

Question Two

Which of the following is the solution to the inequation $\dfrac{x-3}{x} \leq 0$? **1**

(A) $x \leq 3$

(B) $x < 0$ or $x \geq 3$

(C) $0 < x \leq 3$

(D) $0 \leq x \leq 3$

Question Three

Which of the following is the derivative of $2\sin^{-1} 5x$? **1**

(A) $\dfrac{10}{\sqrt{1-25x^2}}$

(B) $\dfrac{1}{\sqrt{1-25x^2}}$

(C) $\dfrac{5}{\sqrt{1-25x^2}}$

(D) $\dfrac{10}{\sqrt{25-x^2}}$

Exam continues next page ...

Question Four

The area under the curve $y = \dfrac{1}{x}$ between $x = 1$ and $x = a$ is 1 square unit. **1**

What is the value of a?

(A) e

(B) 0

(C) $\ln 2$

(D) 1

Question Five

The acute angle between the lines $y = 2x - 5$ and $y = 5x + 3$ is α. **1**

What is the value of $\tan\alpha$?

(A) $\dfrac{3}{11}$

(B) $-\dfrac{3}{11}$

(C) $\dfrac{7}{9}$

(D) $-\dfrac{7}{9}$

Question Six

Suppose A is the point $(1, -2)$ and B is the point $(5, 6)$. The point $P(9, 14)$ divides the interval AB externally in what ratio? **1**

(A) 1:2

(B) 1:1

(C) 3:1

(D) 2:1

Exam continues overleaf ...

Question Seven

What is the domain of $y = \sin^{-1} 2x$? **1**

(A) $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$

(B) $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(C) $-2 \le x \le 2$

(D) $-\frac{1}{2} \le x \le \frac{1}{2}$

Question Eight

What is an expression for $\displaystyle\int \frac{dx}{16 + x^2}$? **1**

(A) $\frac{1}{4} \tan^{-1} 4x + c$

(B) $4 \tan^{-1} \frac{x}{4} + c$

(C) $4 \tan^{-1} 4x + c$

(D) $\frac{1}{4} \tan^{-1} \frac{x}{4} + c$

Question Nine

What is the Cartesian equation of the curve $x = 2\sin\theta, y = 2\cos\theta$? **1**

(A) $x^2 + y^2 = \sqrt{2}$

(B) $x^2 + y^2 = 4$

(C) $x^2 = 4y$

(D) $y^2 = 4x$

Exam continues next page ...

Question Ten

Which of the following functions is a primitive of $\sin^2 x$? **1**

(A) $\frac{1}{2}x - \frac{1}{4}\sin x$

(B) $\frac{1}{2}x - \frac{1}{4}\sin 2x$

(C) $\frac{1}{2}x - \frac{1}{4}\cos x$

(D) $\frac{1}{2}x - \frac{1}{4}\cos 2x$

End of Section I

Exam continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

Question Eleven (15 marks) Use a separate writing booklet. **Marks**

(a) Find $\displaystyle\lim_{x \to 0} \frac{\sin 3x}{x}$. **1**

(b) Find the exact value of $\sin\left(\cos^{-1}(-\frac{3}{5})\right)$. **1**

(c) Evaluate $\displaystyle\int_0^1 \frac{-1}{\sqrt{2-x^2}}\,dx$. **2**

(d) Solve the equation $2\sin^2\theta = \sin\theta$ for $0 \le \theta \le 2\pi$. **2**

(e) (i) Expand $\sin(A - B)$. **1**

(ii) Prove that $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$. **3**

(f) Find the volume of the solid formed when the region bounded by the parabola $y = 4 - x^2$ and the x-axis is rotated about the y-axis. **2**

(g) The volume of a sphere is increasing at a constant rate of $200\,\text{cm}^3/\text{s}$. You are given that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Find the rate of change of the radius, $\dfrac{dr}{dt}$, when $r = 10\,\text{cm}$. Leave your answer in exact form. **3**

Exam continues next page ...

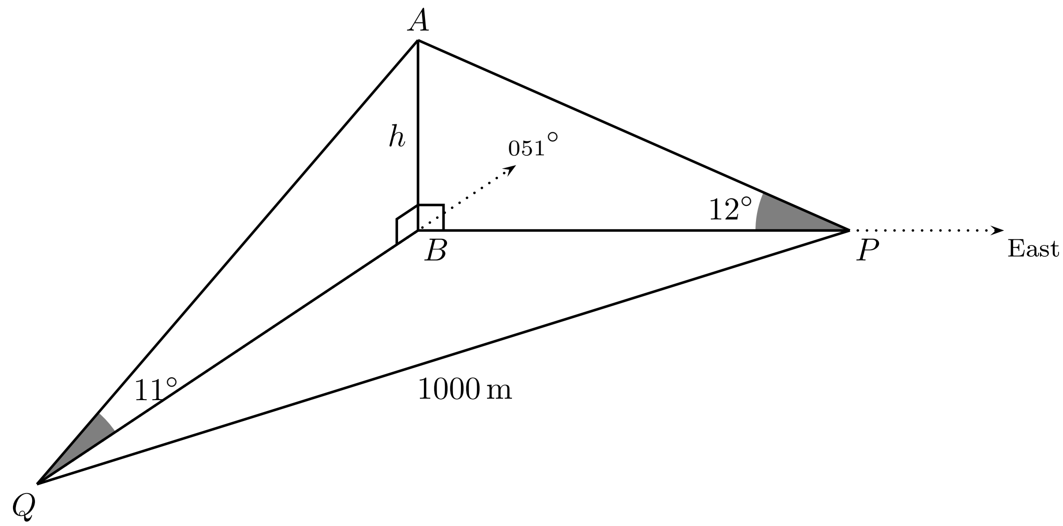
Question Twelve (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Express $\sqrt{3}\cos x - \sin x$ in the form $A\cos(x+\alpha)$, where $A > 0$ and $0 \le \alpha < 2\pi$. **2**

(ii) Write down the maximum value of $\sqrt{3}\cos x - \sin x$. **1**

(iii) Solve the equation $\sqrt{3}\cos x - \sin x = 1$, for $0 \le x \le 2\pi$. **2**

(b)



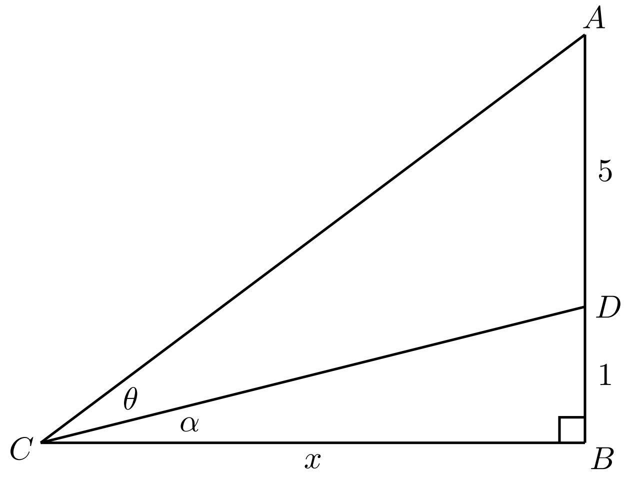
The angle of elevation of a mobile phone tower AB of height h metres from a point P due east of the tower is 12°. From another point Q, the bearing of the mobile phone tower is 051° and the angle of elevation is 11°. The points P and Q are 1000 metres apart and on the same level as the base B of the tower.

(i) Show that $\angle PBQ = 141^\circ$. **1**

(ii) Show that $PB = h\tan 78^\circ$, and write a similar expression for QB. **1**

(iii) Use the cosine rule in $\triangle PBQ$ to calculate h correct to the nearest metre. **2**

(c)



In the diagram above ABC is a triangle with a right angle at B. The point D lies on AB so that AD is 5 units and DB is 1 unit. Let CB be x units. The angle at C is divided into two angles marked θ and α as shown in the diagram.

(i) Show that $\theta = \tan^{-1}\dfrac{6}{x} - \tan^{-1}\dfrac{1}{x}$. **1**

(ii) Show that θ is a maximum when $x = \sqrt{6}$. **3**

(iii) Deduce that the maximum size of $\angle ACD$ is $\theta = \tan^{-1}\dfrac{5\sqrt{6}}{12}$. **2**

Question Thirteen (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the function $f(x) = 2\tan^{-1} x$.

(i) Evaluate $f(\sqrt{3})$. **1**

(ii) Draw the graph of $y = f(x)$, labelling any key features. **2**

(b) Consider the function $f(x) = \dfrac{e^x}{5 + e^x}$.

(i) Show that $f(x)$ has no stationary points. **2**

(ii) Show that $\left(\ln 5, \frac{1}{2}\right)$ is a point of inflexion. **3**

(iii) Find the domain and range of $f(x)$. **1**

(iv) Sketch the curve $f(x) = \dfrac{e^x}{5 + e^x}$, showing any intercepts, asymptotes and points of inflexion. **3**

(v) Explain why $f(x)$ has an inverse function. **1**

(vi) Find the equation of the inverse function $y = f^{-1}(x)$. **1**

(vii) State the domain and range of $y = f^{-1}(x)$. **1**

Exam continues overleaf ...

Question Fourteen (15 marks) Use a separate writing booklet. **Marks**

(a) Find the general solution of $\cos 2x + 3\sin x = 2$. **4**

(b) (i) By considering the sum of an arithmetic series, show that **1**

$$(1 + 2 + 3 + \ldots + n)^2 = \frac{1}{4}n^2(n+1)^2.$$

(ii) By using the Principle of Mathematical Induction prove that **4**

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$$

for all integers $n \geq 1$.

(c) Two distinct points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. You are given $p > q > 0$.

(i) Show that the equation of the tangent to the parabola at P is $y = px - ap^2$. **1**

(ii) The tangents to the parabola at P and Q meet at T. Find the co-ordinates of T. **1**

(iii) The tangents at P and Q intersect at an angle of 45°. Show that $p - q = 1 + pq$. **1**

(iv) Find the equation of the locus of T given that the tangents at P and Q intersect at an angle of 45°. **3**

End of Section II

END OF EXAMINATION

B L A N K P A G E

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \;\; x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax}, \;\; a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \;\; a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\text{NOTE}: \;\ln x = \log_e x, \; x > 0$$

Tear-off pages follow ...

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

NAME: ....

CLASS: ....MASTER: ....

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

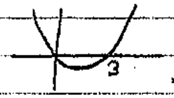
Question Ten

A ◯ B ◯ C ◯ D ◯

Solutions
Form VI Ext 1

Q1. $\frac{\frac{1}{a}-1}{\frac{1}{a}} = \frac{1-a}{1}$ A

Q2 $x(x-3) \le 0$



$x \le 0 \le 3$ C

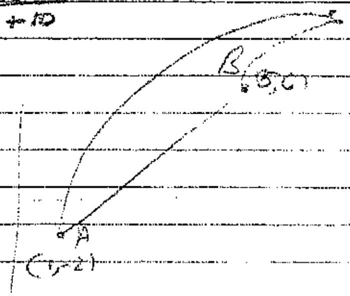
Q3 $y = 2\sin^{-1}5x$
$y' = 2 \times \frac{5}{\sqrt{1-25x^2}}$ A.

Q4.

$\int_1^a \frac{1}{x}dx = \ln x\Big]_1^a = 1$
$= \ln a - \ln 1 = 1$
$= \ln a = 1$
$a = e$ A

Q5.
$\tan\alpha = \frac{2-5}{1+10}$ A

Q6.



D

Q7.
$-1 \le 2x \le 1$
$-\frac{1}{2} \le x \le \frac{1}{2}$ D.

Q8. $a = 4.$ D.

Q9. $x^2+y^2 = 4\sin^2\theta + 4\cos^2\theta$
$= 4$ B.

Q10 B.

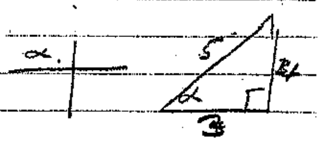
Q11.

(a) $\lim_{x\to 0}\frac{\sin 3x}{x} = 3\lim_{x\to 0}\frac{\sin 3x}{3x}$

$= 3$ ✓ (just need 3).

(b) let $\cos^{-1}(-\frac{3}{5}) = \alpha$, $0 \le \alpha \le \pi$

$\sin\alpha = \frac{4}{5}$ ✓



(c) $\int_0^1 \frac{-1}{\sqrt{2-x^2}}\,dx = \left[\cos^{-1}\frac{x}{\sqrt{2}}\right]_0^1$ ✓

$= \cos^{-1}\frac{1}{\sqrt{2}} - \cos^{-1}0$

$= \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$ ✓ *must* be in radians

(d) $2\sin^2\theta = \sin\theta$

$2\sin^2\theta - \sin\theta = 0$

$\sin\theta(2\sin\theta - 1) = 0$ ✓

$\sin\theta = 0$ or $\sin\theta = \frac{1}{2}$

$\theta = 0, \pi, 2\pi$; rel angle is $\frac{\pi}{6}$

$\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$\theta = 0, \pi, 2\pi, \frac{\pi}{6}$ or $\frac{5\pi}{6}$. ✓ (need all for 2nd mk).

(e) (i) $\sin(A-B) = \sin A\cos B - \cos A\sin B$ ✓

(ii) $LHS = \frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta}$

$= \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$ ✓

$= \frac{\sin(3\theta-\theta)}{\sin\theta\cos\theta}$

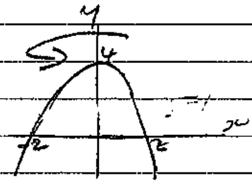
$= \frac{\sin 2\theta}{\sin\theta\cos\theta}$ ✓

$= \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$ ✓

$= 2$

$= RHS.$

(f)



$V = \pi\int_0^4 x^2\,dy$

$= \pi\int_0^4 (4-y)\,dy$ ✓

$= \pi\left[4y - \frac{1}{2}y^2\right]_0^4$

$= \pi((16-8)-(0))$

$= 8\pi\ u^3$ ✓

(g) Now $\frac{dr}{dt} = \frac{dr}{dV}\cdot\frac{dV}{dt}$ ✓

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dV} = \frac{1}{4\pi r^2}$ ✓

$= \frac{1}{4\pi r^2} \times 200$

$= \frac{1}{4\pi\, 100} \times 200$

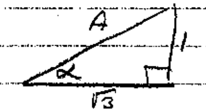
$= \frac{1}{2\pi}\ cm\,s^{-1}$ ✓

(There are other ways to do this).

Q12.

(a) (i) $\sqrt{3}\cos x - \sin x = A\cos(x+\alpha)$
$= A\cos x\cos\alpha - A\sin x\sin\alpha$

So $\sqrt{3} = A\cos\alpha$ and $1 = A\sin\alpha$.



$\Rightarrow A^2 = 1+3 = 4$
$A = 2$ ✓

$\Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$
$\alpha = \frac{\pi}{6}$ ✓

So $\sqrt{3}\cos x - \sin x = 2\cos(x+\frac{\pi}{6})$

(ii) 2 ✓

(iii) $\sqrt{3}\cos x - \sin x = 2\cos(x+\frac{\pi}{6}) = 1$
$\cos(x+\frac{\pi}{6}) = \frac{1}{2}$ ✓

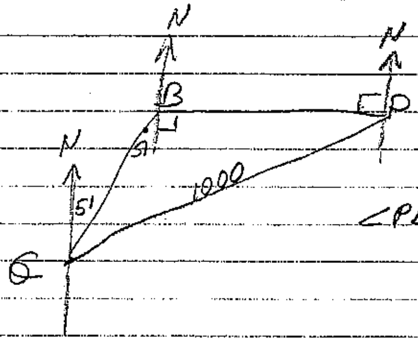
related angle is $\frac{\pi}{3}$.

so $x+\frac{\pi}{6} = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ $\frac{10\pi}{6}$ $\frac{9\pi}{6}$

$x = \frac{\pi}{6}$ or $\frac{3\pi}{2}$ ✓

(b)

(i)

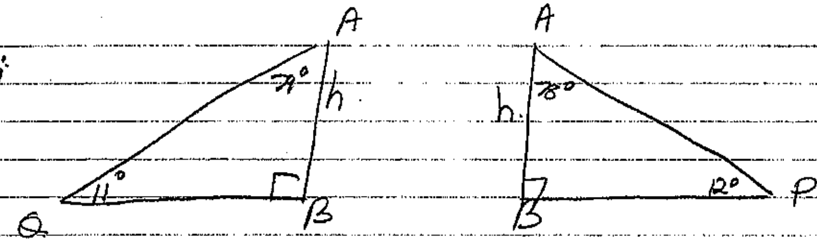


$\angle PBQ = 51° + 90°$ ✓
$= 141°$

just need to show 51° + 90°.

(ii) over

(ii)



Using $\triangle ABP$, $\tan 78° = \frac{BP}{h}$ ✓

so $BP = h\tan 78°$

Similarly $QB = h\tan 79°$ — if they don't write this here — no matter they will have to use it in (iii).

(iii)

$QP^2 = QB^2 + PB^2 - 2\times QB\times PB\times\cos B$

$1000^2 = h^2\tan^2 79° + h^2\tan^2 78 - 2\times h^2\tan 79\tan 78 \times\cos 141°$ ✓

$h^2 = \frac{1000^2}{\tan^2 79° + \tan^2 78 - 2\tan 79°\tan 78°\cos 141°}$

$= \frac{1000^2}{86.21855}$

$= 11598$

$h \doteqdot 107.69$
$\doteqdot 108$ m ✓

(e)

(i) $\angle ACD = \angle ACB - \angle DCB$ ✓

$$\theta = \tan^{-1}\frac{6}{x} - \tan^{-1}\frac{1}{x}$$

we need them to identify $\angle ACB$ as $\tan^{-1}\frac{6}{x}$ and $\angle DCB$ as $\tan^{-1}\frac{1}{x}$ in some way.

(ii)
$$\theta = \tan^{-1}\frac{6}{x} - \tan^{-1}\frac{1}{x}$$

$$\frac{d\theta}{dx} = \frac{1}{1+\frac{36}{x^2}} \times \left(\frac{-6}{x^2}\right) - \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right) = 0 \text{ at stat pt.}$$

$$\frac{-6}{x^2+36} + \frac{1}{1+x^2} = 0$$ ✓

$$\frac{6}{x^2+36} = \frac{1}{1+x^2}$$

$$6 + 6x^2 = x^2 + 36$$
$$5x^2 = 30$$
$$x^2 = 6$$ ✓
$$x = \pm\sqrt{6}$$
$x = \sqrt{6}$, $x > 0$ since it is a length.

Check for maximum

x	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$
$\frac{d\theta}{dx}$	$\frac{-6}{41} + \frac{1}{6}$ 0.02 +ve	0	$\frac{-6}{43} + \frac{1}{8}$ −0.01 −ve

✓

/ — \

So we have max θ for $x = \sqrt{6}$

(iii) $x = \sqrt{6}$.

$$\theta = \tan^{-1}\frac{6}{\sqrt{6}} - \tan^{-1}\frac{1}{\sqrt{6}}$$

$$\tan\theta = \frac{\frac{6}{\sqrt{6}} - \frac{1}{\sqrt{6}}}{1 + \frac{6}{\sqrt{6}}\cdot\frac{1}{\sqrt{6}}}$$ ✓

$$= \frac{\frac{5}{\sqrt{6}}}{1+1}$$

$$= \frac{5}{2\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$$ ✓

$$= \frac{5\sqrt{6}}{12}$$

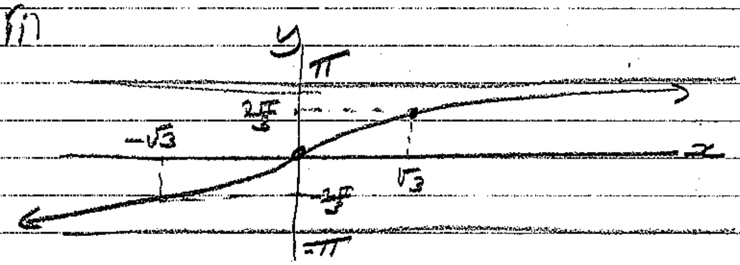
so $\theta = \tan^{-1}\frac{5\sqrt{6}}{12}$

Q13.

(a) (i) $f(\sqrt{3}) = 2\tan^{-1}\sqrt{3}$

$= \frac{2\pi}{3}$ ✓

(ii)



✓ asymptotes

✓ shape.

(b) (i) $f(x) = \frac{e^x}{5+e^x}$

$f'(x) = \frac{(5+e^x)e^x - e^x e^x}{(5+e^x)^2}$ ✓

$= \frac{5e^x}{(5+e^x)^2} \neq 0$ ✓ since $5e^x > 0$.

So no stationary points

(ii) $f''(x) = \frac{(5+e^x)^2 5e^x - 5e^x 2(5+e^x)e^x}{(5+e^x)^4}$

$= \frac{5e^x(5+e^x)[5+e^x - 2e^x]}{(5+e^x)^4}$

$= \frac{5e^x(5-e^x)}{(5+e^x)^3}$ ✓

$f''(x) = 0$ at a possible pt of inflection

$5e^x(5-e^x) = 0$

$e^x = 5$ ✓

$x = \ln 5$

$y = \frac{e^{\ln 5}}{5+e^{\ln 5}}$

$= \frac{5}{10}$

$= \frac{1}{2}$.

check for concavity change.

x	$\ln 4$	$\ln 5$	$\ln 6$.
$f''(x)$	$\frac{20.9.1}{9^2}$ +ve ∪	0	$\frac{30.11.(-1)}{11^2}$ −ve ∩

✓

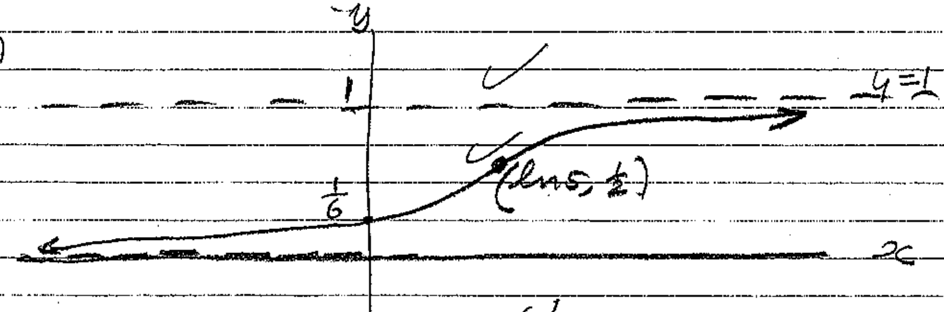
We have concavity change

So, $(\ln 5, \frac{1}{2})$ is a point of inflection

(iii) D: all x ✓ need both for mark

R: $0 < y < 1$

(iv)



✓ shape

(v) f(x) has an inverse because a horizontal line cuts it once only. ✓

(or any good reason) e.g. the function is increasing for all x.

(vi)
$$y = \frac{e^x}{5+e^x}$$
$$x = \frac{e^y}{5+e^y}$$
$$5x + xe^y = e^y$$
$$e^y(1-x) = 5x$$
$$e^y = \frac{5x}{1-x}$$
$$y = \ln\left(\frac{5x}{1-x}\right).$$ ✓

(vii) D: $0 < x < 1$ ✓

R: all y.

✓ if (vii) corresponds with (iii)

Q14.

(a) $\cos 2x + 3\sin x = 2$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$

$1 - 2\sin^2 x + 3\sin x = 2$ ✓

$2\sin^2 x - 3\sin x + 1 = 0$

$(2\sin x - 1)(\sin x - 1) = 0$ ✓

$\sin x = \frac{1}{2}$ or $\sin x = 1$

related angle is $\frac{\pi}{6}$ ✓/✓ $\quad x = 2n\pi + \frac{\pi}{2}$ ✓

$x = n\pi + (-1)^n \frac{\pi}{6}$, n an integer

✓

Note there are many other correct ways to express these answers.
Accept answers in degrees.

(b) (i) $1+2+3+\cdots n = \frac{n}{2}(1+n)$ ✓

so $(1+2+3+\cdots n)^2 = \frac{n^2}{4}(1+n)^2$

(ii)
A: Consider $n=1$. ✓

$LHS = 1^3 = 1$

$RHS = 1^2 = 1$

So the statement is true for $n=1$.

B: Suppose the statement is true for some integer k, $k > 1$

ie suppose $1^3 + 2^3 + \cdots + k^3 = (1+2+\cdots+k)^2$

and show ~~that~~ $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (1+2+\cdots+k+(k+1))^2$

Now $1^3+2^3+3^3+\cdots k^3+(k+1)^3$

$= (1+2+3+\cdots k)^2 + (k+1)^3$, using the induction hypothesis ✓

$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$, using (i) ✓

$= \frac{(k+1)^2}{4}(k^2+4k+4)$

$= \frac{1}{4}(k+1)^2(k+2)^2$ ✓

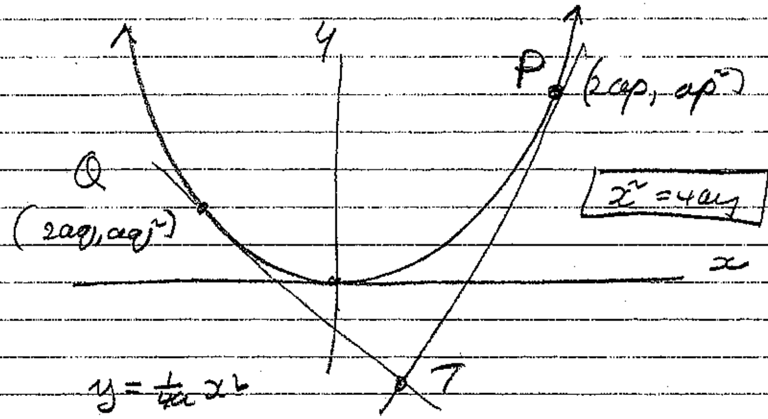
$= \frac{1}{4}(k+1)^2((k+1)+1)$

$= (1+2+3+\cdots(k+1))^2$ using (i)

C: So, by Step A&B and Mathematical induction the given statement is true

(need the last statement for full marks)

(c)



(i) $y = \frac{1}{4a}x^2$

$\frac{dy}{dx} = \frac{x}{2a}$.

$x = 2ap$, $m = \frac{2ap}{2a} = p$. ✓

So tangent is $y - ap^2 = p(x - 2ap)$.

$y = px - ap^2$.

(ii) $y = px - ap^2$

$y = qx - aq^2$

$qx - aq^2 = px - ap^2$

$px - qx = ap^2 - aq^2$

$x(p-q) = a(p+q)(p-q)$ $\quad p \neq q$

$x = a(p+q)$

$y = ap(p+q) - ap^2$ ✓

$= apq$

T is $(a(p+q), apq)$

(iii) $\tan 45^\circ = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$1 = \frac{p-q}{1+pq}$, $p > q > 0$.

$1 + pq = p - q$.

(iv) At T, $x = a(p+q)$, $y = apq$

$\frac{x}{a} = p+q$ $\qquad$ $\frac{y}{a} = pq$

now $(p-q)^2 = (p+q)^2 - 4pq$.

so $(1+pq)^2 = \frac{x^2}{a^2} - \frac{4y}{a}$ (using iii)

$\left(1 + \frac{y}{a}\right)^2 = \frac{x^2}{a^2} - \frac{4y}{a}$

$(a+y)^2 = x^2 - 4ay$

$a^2 + 2ay + y^2 = x^2 - 4ay$

locus is $a^2 + 6ay + y^2 - x^2 = 0$